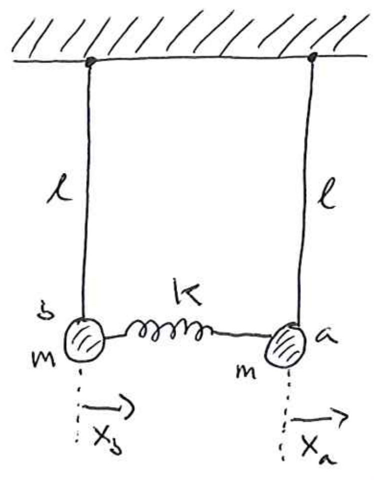
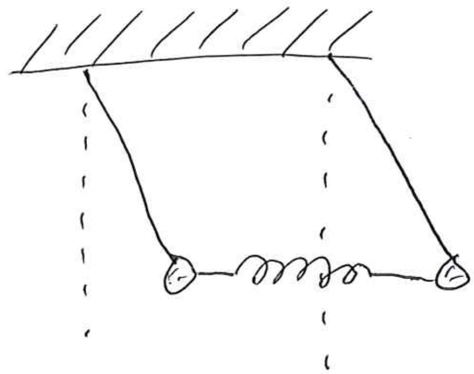
Coupled Oscillators [King Ch. 4]

- Single (isolated) harmonic oscillators have only one way of oscillating, and the do so at their natural frequency ω_0
- Multiple oscillators that are coupled together (meaning energy can be exchanged b/t each osc.) collectively exhibit multiple frequencies of oscillation.
  - → each frequency relates to a different way the coupled system can oscillate
  - → These different ways are called "normal modes" and the associated frequencies are "normal frequencies"
- Normal modes characterized by the fact that all parts of the system oscillate at the same frequency (but not necessarily the same phase)
- Important to understand b/c harmonic oscillators rarely exist in isolation, & there are many examples of coupled oscillators
  - rattling car
  - vibrating atoms in crystals
  - ∞ coupled oscillators → continuous media & WAVES

Example: Consider two pendulums connected by a spring w/ spring constant k



• Case (A): Displace both masses by same amount in same direction



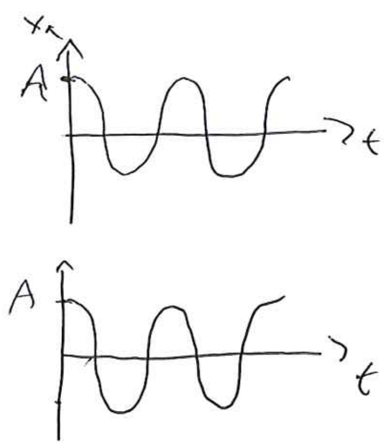
• spring remains unstretched

k plays no role

• ~~system~~ pendulums moves as if unconnected

$\boxed{x_a = A\cos\omega_1 t}$ $\boxed{x_b = A\cos\omega_1 t}$

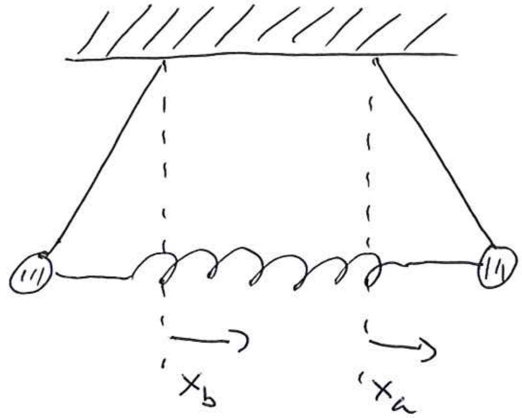
$\boxed{\omega_1 = \sqrt{g/\ell}}$ ($\phi_a = \phi_b = 0$ b/c start from rest)



→ the masses oscillate in phase

"First normal mode of oscillation"

Case (B): Displace both masses by same amount in <u>opposite</u> directions



· spring gets stretched and compressed as pendulums swing

→ additional restoring force!

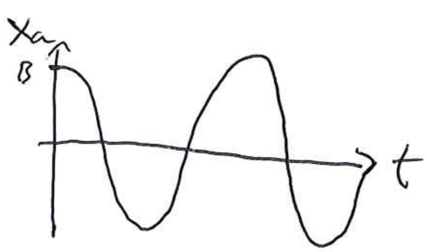
→ $x_a = -x_b$ due to mirror symmetry of setup

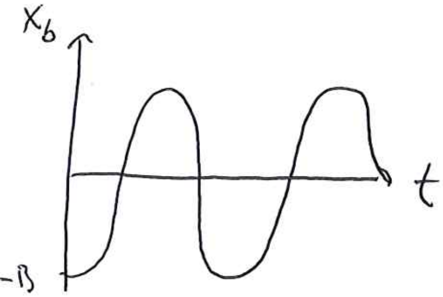
· Eq. of motion for mass a:

$$m\frac{d^2x_a}{dt^2} = \underbrace{-\frac{mgx_a}{\ell}}_{\substack{\text{restoring}\\ \text{force from}\\ \text{gravity}}} \underbrace{- 2kx_a}_{\substack{\text{restoring}\\ \text{force from}\\ \text{spring}}} = -m\omega_2^2 x_a \quad \text{w/} \quad \omega_2 = \sqrt{g/\ell + \frac{2k}{m}}$$

Solution: $x_a = B\cos\omega_2 t$

$x_b = -x_a = -B\cos\omega_2 t$





→ masses oscillate 180° <u>out-of-phase!</u>

"2^{nd} normal mode of oscillation"

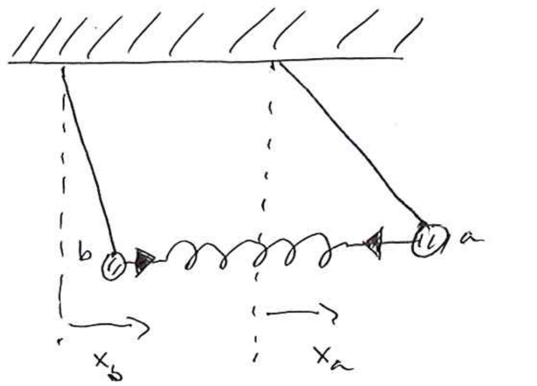
$\omega_2 > \omega_1$ b/c has increased restoring force

Summary of normal mode characteristics

- masses oscillate w/ <u>same</u> frequency
- masses perform simple harmonic motion w/ constant amplitude
- well-defined phase difference b/t masses: 0° or 180°
- if system gets excited in normal mode, it stays in that mode

Superposition of normal modes

- The above cases are special in that the motion was confined to a single normal mode: either $x_a = x_b$ or $x_a = -x_b$ at all times
- Generally, coupled oscillators can be in a <u>superposition</u> of the two normal modes



- Here, you can see that $x_a \neq \pm x_b$ at a given snapshot in time
- This gives spring extension $(x_a - x_b)$ that produces a force/tension $F = k(x_a - x_b)$

- Restoring force on mass a is then:

$$-\frac{mgx_a}{\ell} - \underbrace{k(x_a - x_b)}$$

→ stretched spring gives force pushing mass a <u>toward</u> EP

" " mass b:

$$-\frac{mgx_b}{\ell} + \underbrace{k(x_a - x_b)} \cancel{-k(x_b - x_a)}$$

stretched spring pulls mass b <u>away</u> from EP

$\Rightarrow$ eqs. of motion:

$$\left.\begin{array}{ll} \text{[eq. 4.6]} & \dfrac{d^2 x_a}{dt^2} + \dfrac{g x_a}{\ell} + \dfrac{k}{m}(x_a - x_b) = 0 \\[2ex] \text{[eq 4.7]} & \dfrac{d^2 x_b}{dt^2} + \dfrac{g x_b}{\ell} - \dfrac{k}{m}(x_a - x_b) = 0 \end{array}\right\}$$

x_a & x_b appear in both eqs.

$\rightarrow$ means they are coupled!

$\rightarrow$ These eqs. cannot be solved separately, must be solved simultaneously

· To do so, take the sum & difference:

Sum: $[4.6] + [4.7]$: $\dfrac{d^2(x_a + x_b)}{dt^2} + \dfrac{g(x_a + x_b)}{\ell} = 0$

$\rightarrow$ SHO w/ variable $(x_a + x_b)$ and freq $\underbrace{\omega_1 = \sqrt{g/\ell}}_{\text{1st normal mode!}}$

Difference: $[4.6] - [4.7]$: $\dfrac{d^2(x_a - x_b)}{dt^2} + \left(\dfrac{g}{\ell} + \dfrac{2k}{m}\right)(x_a - x_b) = 0$

$\rightarrow$ SHO w/ variable $(x_a - x_b)$ & $\underbrace{\omega_2 = \sqrt{g/\ell + \frac{2k}{m}}}_{\text{2nd normal mode!}}$

· Now define new coordinates that decouple diff. eqs.

$\boxed{q_1 = (x_a + x_b)}$ & $\boxed{q_2 = (x_a - x_b)}$ "normal coordinates"

$\rightarrow$ these oscillate <u>independently</u> (ie, not coupled) w/ freq. ω_1 ~~&~~ and ω_2 "normal frequencies"

$$\boxed{\frac{d^2 q_1}{dt^2} + \omega_1^2 q_1 = 0}, \quad \boxed{\frac{d^2 q_2}{dt^2} + \omega_2^2 q_2 = 0}$$

· General solutions are:

$$\boxed{q_1 = C_1 \cos(\omega_1 t + \phi_1)}, \quad \boxed{q_2 = C_2 \cos(\omega_2 t + \phi_2)}$$

· Special cases:

(A) $q_2 = 0 \;\rightarrow\; x_a = x_b$ at all times! $\quad (x_a - x_b = 0)$

(B) $q_1 = 0 \;\rightarrow\; x_a = -x_b$ at all times! $\quad (x_a + x_b = 0)$

→ So from the general case we recover our initial consideration

· This means that the motion of masses can generally be expressed as a superposition of normal modes:

$$x_a = \frac{1}{2}(q_1 + q_2)$$
$$x_b = \frac{1}{2}(q_1 - q_2)$$

Constants C_1, C_2, ϕ_1, ϕ_2 determined by initial conditions (displacement and velocity)

~~(if initially at rest, $\phi_1 = \phi_2 = 0$)~~

· If two masses start from rest, then $\phi_1 = \phi_2 = 0$ ~~and~~

//

Energy

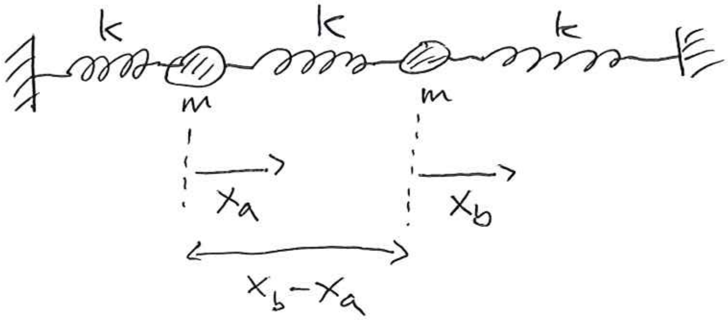
$$E = \underbrace{\frac{1}{2} m \left(\frac{dx_a}{dt}\right)^2 + \frac{1}{2} m \left(\frac{dx_b}{dt}\right)^2}_{KE} + \underbrace{\frac{1}{2} \frac{mg}{\ell}(x_a^2 + x_b^2)}_{PE\ \text{pendulum}} + \underbrace{\frac{1}{2}k(x_a - x_b)^2}_{PE\ \text{spring}}$$

· in terms of q_1, q_2:

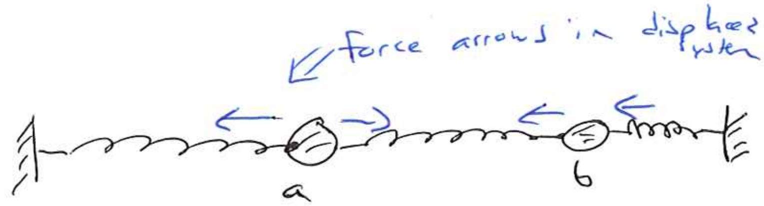
$$\rightarrow E = \left[\frac{1}{4} m \left(\frac{dq_1}{dt}\right)^2 + \frac{1}{4}\left(\frac{mg}{\ell}\right) q_1^2\right] + \left[\frac{1}{4} m \left(\frac{dq_2}{dt}\right)^2 + \frac{1}{4}\left(\frac{mg}{\ell} + 2k\right) q_2^2\right]$$

→ 2 independent oscillators → no energy flow b/t them

Oscillating masses coupled by springs



· Consider forces for each mass:



$$m\frac{d^2x_a}{dt^2} = -kx_a + k(x_b - x_a) = kx_b - 2kx_a \qquad [\text{eq. } 4.18]$$

+ b/c if $x_b - x_a > 0$, it creates force pulling x_a away from EP, not toward

$$m\frac{d^2x_b}{dt^2} = -kx_b - k(x_b - x_a) = kx_a - 2kx_b \qquad [\text{eq. } 4.19]$$

· We are looking for normal mode solutions of form

$$x_a = A\cos\omega t \quad , \quad x_b = B\cos\omega t$$

both masses oscillate at same frequency

· Substituting x_a in eq. 4.18

$$-Am\omega^2\cos\omega t = kB\cos\omega t - 2kA\cos\omega t$$

gives $\rightarrow$ $$\frac{A}{B} = \frac{k}{(2k - m\omega^2)} \qquad [4.20]$$

· Substitute x_b in eq. 4.19

$$-Bm\omega^2\cos\omega t = kA\cos\omega t - 2kB\cos\omega t$$

$$\rightarrow \frac{A}{B} = \frac{(2k - m\omega^2)}{k}$$

- Thus, $\dfrac{A}{B} = \dfrac{(2k - m\omega^2)}{k} = \dfrac{k}{(2k - m\omega^2)}$

$\rightarrow (2k - m\omega^2)^2 = k^2$

$\rightarrow (2k - m\omega^2) = \pm k$

$$\Rightarrow \underbrace{\omega^2 = \frac{k}{m} \quad \text{or} \quad \frac{3k}{m}}$$

These are the two normal frequencies

- Plug $\omega^2 = \frac{k}{m}$ into $\overset{[4.20]}{\dfrac{A}{B} = \dfrac{k}{2k - m\omega^2}}$ gives $A = B$

$\rightarrow$ 1st normal mode, both masses oscillate in same direction

$x_a = x_b = A\cos\omega_1 t \qquad \omega_1 = \sqrt{\frac{k}{m}}$

- Plug $\omega^2 = \frac{3k}{m}$ in [4.20] gives $A = -B$

$x_a = A\cos\omega_2 t \qquad x_B = -A\cos\omega_2 t \qquad \omega_2 = \sqrt{\frac{3k}{m}}$

$\rightarrow$ 2nd normal mode, masses oscillate in opposite directions

- As before general solution is $x_a = C_1\cos\omega_1 t + C_2\cos\omega_2 t$

$x_b = C_1\cos\omega_1 t - C_2\cos\omega_2 t$

(include ϕ_1, ϕ_2 phases if initial velocity at $t=0$ is nonzero)

Matrix Formalism for normal modes

- In our derivation above we had 2 eqs. to solve of form:

$$\frac{2k}{m}A - \frac{k}{m}B = \omega^2 A$$

$$-\frac{k}{m}A + \frac{2k}{m}B = \omega^2 B$$

$$\xrightarrow{\text{matrix}} \begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \omega^2 \begin{pmatrix} A \\ B \end{pmatrix}$$

"eigen value problem" $\rightarrow \begin{pmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$

"eigen vector"

- Nonzero solutions if and only iff determinat = 0

$$\rightarrow \left(\frac{2k}{m} - \omega^2\right)^2 - \left(-\frac{k}{m}\right)^2 = 0 \quad \Rightarrow \omega_1 = \sqrt{k/m} \quad \omega_2 = \sqrt{3k/m}$$

→ very useful if more than two oscillators involved

→ linear algebra, used a lot in quantum class

Outlook:

- Here, 2 masses coupled together moving in one dimension
  → 2 normal modes w/ coordinates q_1, q_2 and freq. ω_1, ω_2

- General: N masses connected, moving in 3D
  → 3N normal modes with $q_1, \dots q_{3N}$ and $\omega_1 \dots \omega_{3N}$

Ex: Atoms in crystal are coupled

- if one atom vibrates, all atoms vibrate b/c coupled
- crystal w/ N atoms has 3N normal modes!

→ Debye model for temp. dep. of heat capacity in crystals! (also need quantum harmonic osc.)